TRANSIENT HEAT CONVECTION WITHIN A SPHERICAL FILM

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PMM Vol.22, No.3, 1958, pp.419-423

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(Received 25 April 1955)

An approximate solution is obtained for the problem of weak transient heat convection within fluid trapped between two concentric spherical walls, which are maintained at constant temperature. The assumption is that at the instant of starting, the fluid is at rest, and is at a uniform temperature everywhere but differing from that of the walls. The temperature and velocity of the fluid are found to the zero and first approximations, expressed as expansions of these terms in powers of the Rayleigh number. Finally, in these approximations, the effect of convection on the rate of cooling of the fluid is reviewed.

1. Statement of Problem. We will deal with the problem of transient heat convection in fluid confined within the space bounded by two concentric spheres of radius R_1 and R_2 ($R_1 < R_2$), maintained at constant temperature T_1, if, at the initial instant the fluid was at rest and was at constant temperature $T_2 \neq T_1$.

We will make use of the well known equations of free convection [1],

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla\frac{p'}{\rho} + \nu\Delta\mathbf{v} - \beta\mathbf{g}T', \qquad \frac{\partial T'}{\partial t} + (\mathbf{v}\nabla T') = \chi\Delta T', \qquad \operatorname{div}\mathbf{v} = 0 \qquad (1.1)$$

where $\mathbf{v}$ is the velocity of the fluid, T' and p', temperature and pressure reckoned from the values which they would have had under conditions of mechanical and thermal equilibrium at some mean temperature, ρ is the mean density of the liquid, ν, β, χ, respectively, are the coefficients of kinematic viscosity, thermal expansion and thermal conductivity of the fluid, and g is the acceleration due to gravity.

We eliminate p' by applying operation rot to (1.1); thus

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{v} + \nu\operatorname{rot}\operatorname{rot}\operatorname{rot}\mathbf{v} - \operatorname{rot}[\mathbf{v}\operatorname{rot}\mathbf{v}] = \beta[\mathbf{g}\nabla T'] \qquad (1.2)$$

We introduce the dimensionless variables,

$$\mathbf{u} = \frac{l}{\chi}\mathbf{v}, \qquad \tau = \frac{T'}{T_2 - T_1}, \qquad \mathbf{r}_1 = \frac{\mathbf{r}}{l}, \qquad t_1 = \frac{\nu}{l^2}t \tag{1.3}$$

where $l = R_2 - R_1$. Using this notation for the nondimensional time and coordinates in what follows, we arrive at the following dimensionless equations of the problem:

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{u} + \operatorname{rot}\operatorname{rot}\operatorname{rot}\mathbf{u} - \frac{1}{N_{\mathrm{Pr}}}\operatorname{rot}[\mathbf{u}\operatorname{rot}\mathbf{u}] = N_{\mathrm{Pr}}N_{\mathrm{Gr}}[\nabla\tau\mathbf{k}] \tag{1.4}$$

$$\triangle\tau = (\mathbf{u}\nabla\tau) + N_{\mathrm{Pr}}\frac{\partial\tau}{\partial t}, \qquad \operatorname{div}\mathbf{u} = 0, \qquad \left(N_{\mathrm{Pr}} = \frac{\nu}{\chi}\right) - \text{(Prandtl No)}$$

$$N_{\mathrm{Gr}} = -\frac{g\beta(T_2 - T_1)l^3}{\nu^2} - \text{(Grashof No)} \qquad \mathbf{k} = -\frac{\mathbf{g}}{g} \tag{1.5}$$

If we reckon the value T of the fluid temperature from T_1, then $T' = T - T_1$ and $\tau = (T - T_1)/(T_2 - T_1)$. It follows that the temperature τ at the boundaries of the field is zero at any given instant of time, whilst at the initial instant it is unity:

$$\tau(\Gamma, t) = 0, \qquad \tau(\mathbf{r}, 0) = 1 \tag{1.6}$$

The velocity of the fluid $\mathbf{u}$ at the boundary is taken to be zero at all times, whilst at the initial instant it is zero over the whole fluid as stated for the problem at the outset:

$$\mathbf{u}(\Gamma, t) = \mathbf{u}(\mathbf{r}, 0) = 0 \tag{1.7}$$

Assuming the convection to be weak the required values can be evaluated from (1.4) as a power series in Grashof numbers [2]. As the Grashof number only enters the equations of the problem in the combination called the Rayleigh number, the quantities we are looking for can be expanded in a series of Rayleigh number:

$$\mathbf{u} = \mathbf{u}_1(N_{\mathrm{Pr}}N_{\mathrm{Gr}}) + \mathbf{u}_2(N_{\mathrm{Pr}}N_{\mathrm{Gr}})^2 + \ldots, \quad \tau = \tau_0 + \tau_1(N_{\mathrm{Pr}}N_{\mathrm{Gr}}) + \tau_2(N_{\mathrm{Pr}}N_{\mathrm{Gr}})^2 + \ldots \tag{1.8}$$

If we limit ourselves to the zero and first Rayleigh Number approximations, we obtain the following equations of the problem;

$$\triangle\tau_0 = N_{\mathrm{Pr}}\frac{\partial\tau_0}{\partial t}, \qquad \triangle\tau_1 = N_{\mathrm{Pr}}\frac{\partial\tau_1}{\partial t} + (\mathbf{u}_1\nabla\tau_0) \tag{1.9}$$

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{u}_1 + \operatorname{rot}\operatorname{rot}\operatorname{rot}\mathbf{u}_1 = [\nabla\tau_0\mathbf{k}], \qquad \operatorname{div}\mathbf{u}_1 = 0 \tag{1.10}$$

2. Zero Approximation. It is evident that the zero approximation, equation (1.9), describes the process of cooling of the heated fluid by molecular heat conduction(to be explicit T_2 is taken $> T_1$).

It follows from symmetry that $\tau_0 = \tau_0(r, t)$, where r is the dimensionless distance from the centre of the sphere.

Using spherical coordinates the first equation (1.9) is as follows

$$\frac{\partial^2}{\partial r^2}(r\tau_0) = N_{\mathrm{Pr}}\frac{\partial}{\partial t}(r\tau_0) \tag{2.1}$$

In order to be consistent with the postulated initial and boundary conditions, we require that the solution of (2.1) satisfies the following conditions:

$$\tau_0(r_1,\ t) = \tau_0(r_2,\ t) = 0, \qquad \tau_0(r,\ 0) = 1 \qquad \left(r_1 = \frac{R_1}{l},\quad r_2 = \frac{R_2}{l} = r_1 + 1\right)$$

Solving by the method of Fourier, we get

$$r\tau_0 = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2}{N_{\mathrm{Pr}}}t\right)\sin n\pi(r - r_1) \qquad \left(A_n = \frac{2}{n\pi}(r_1 - r_2\cos n\pi)\right) \tag{2.2}$$

3. First Approximation. 1. We will first find the velocity of the fluid. To do this we have to solve equations (1.10) taking (2.2) into account with the corresponding initial and boundary conditions.

Note that the fluid motion is symmetrical with respect to the vertical diameter of the sphere. We will introduce the dimensionless stream function $\psi = \psi(r,\ \theta,\ t)$. For the components of the velocity $\mathbf{u}_1$ in the spherical system, we will have:

$$u_{1r} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_{1\theta} = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{3.1}$$

We will look for a stream function ψ in the form

$$\psi = f(r,\ t)\sin^2\theta \tag{3.2}$$

Now putting (3.2) into (3.1), and the result of this into (1.10), we obtain

$$\frac{\partial^2}{\partial r^2}\left(\frac{2f}{r^2} - \frac{\partial^2 f}{\partial r^2}\right) - \frac{\partial}{\partial t}\left(\frac{2f}{r^2} - \frac{\partial^2 f}{\partial r^2}\right) - \frac{2}{r^2}\left(\frac{2f}{r^2} - \frac{\partial^2 f}{\partial r^2}\right) = r\frac{\partial \tau_0}{\partial r} \tag{3.3}$$

From conditions (1.7) we get the following conditions:

$$f(r_1,\ t) = f(r_2,\ t) = 0, \qquad f'(r_1,\ t) = f'(r_2,\ t) = 0, \qquad f(r,\ 0) = f'(r,\ 0) = 0 \tag{3.4}$$

(both here, and in what follows, the prime denotes differentiation with respect to r). We will look for a solution to the nonhomogeneous linear differential equation (3.5) in the following form:

$$f(r,\ t) = \sum_{k=1}^{\infty} R_k(r)\, S_k(t) \tag{3.5}$$

where

$$R_k(r) = \frac{\sqrt{r}}{\lambda_k^2}\left[C_k^{(1)} J_{3/2}(\lambda_k r) + C_k^{(2)} J_{-3/2}(\lambda_k r)\right] + C_k^{(3)} r^2 + \frac{C_k^{(4)}}{r} \quad (k = 1,2,\ldots) \tag{3.6}$$

are eigen functions of the homogeneous problem corresponding to the nonhomogeneous problem (3.3), (3.4), whilst $S_k(t)$ are, so far, undetermined functions.

In equation (3.6) $\lambda_k (k = 1,2..)$ are roots of the equation

$$3\left(r_1^2 + r_2^2\right)\cos\lambda + \left(r_2^3 - r_1^3\right)\lambda\sin\lambda - 3\,\frac{\sin\lambda}{\lambda} = 6r_1r_2 \tag{3.7}$$

and the constants C_k have the values

$$\begin{gathered}
C_k^{(1)} = (r_1r_2)^{1/2}\left[\sqrt{r_2}\,J_{-1/2}(\lambda_k r_1) - \sqrt{r_1}\,J_{-1/2}(\lambda_k r_2)\right] \\
C_k^{(2)} = (r_1r_2)^{1/2}\left[\sqrt{r_2}\,J_{1}\ (\lambda_k r_1) - \sqrt{r_1}\,J_{1/2}(\lambda_k r_2)\right] \\
C_k^{(3)} = \frac{(r_1r_2)^{1/2}}{3\lambda_k}\left[J_{1/2}(\lambda_k r_1)\,J_{-1/2}(\lambda_k r_2) - J_{-1/2}(\lambda_k r_1)\,J_{1/2}(\lambda_k r_2)\right] \\
C_k^{(4)} = -\,C_k^{(3)} r_1^3 - \frac{r_1^{3/2}}{\lambda_k^2}\left[C_k^{(1)} J_{3/2}(\lambda_k r_1) + C_k^{(2)} J_{-3/2}(\lambda_k r_1)\right]
\end{gathered} \tag{3.8}$$

(J is the Bessel function). We may note that the system of functions $R_k(r)$ $(k = 1, 2, \ldots)$ is complete (see Ref. [3]).

It can be seen on checking that the functions $R_k(r)$ and $(2R_m/r^2 - R_m'')$ for $k \neq m$ are mutually orthogonal and therefore if we multiply both sides of (3.5) by $(2R_k/r^2 - R_k)$ and integrate with respect to r within the limits r_1 and r_2 we obtain

$$\int_{r_1}^{r_2} f\left(\frac{2R_k}{r^2} - R_k''\right)dr = S_k(t)\int_{r_1}^{r_2} R_k\left(\frac{2R_k}{r^2} - R_k''\right)dr = \frac{S_k(t)}{\lambda_k^2}\int_{r_1}^{r_2}\left(\frac{2R_k}{r^2} - R_k''\right)^2 dr \tag{3.9}$$

To find the integral on the left-hand side of (3.9) we multiply both sides of (3.3) by R_k and integrate with respect to r from r_1 to r_2 [4]. Integrating by parts we shift the differentiating operation from the function f to R_k, and, after straightforward rearrangement we obtain the equation

$$\frac{d}{dt}G_k + \lambda_k^2 G_k = -H_k \qquad \left(G_k(t) = \int_{r_1}^{r_2}\left(\frac{2R_k}{r^2} - R_k''\right) f\,dr, \qquad H_k(t) = \int_{r_1}^{r_2} R_k r\,\frac{\partial \tau_0}{\partial r}\,dr\right) \tag{3.10}$$

Now we solve the equation (3.10) for $G_k(t)$, taking the initial conditions (3.4) into account. The result of the calculation is

$$G_k(t) = -\exp\left(-\lambda_k^2 t\right)\int_0^t H_k(t)\exp\left(\lambda_k^2 t\right)dt \tag{3.11}$$

This expression for the integral is put into (3.9) and the second equation there is solved for $S_k(t)$. Introducing the value of $S_k(t)$ so obtained into (3.5), we then arrive at a solution of equation (3.3) which formally satisfies all the conditions (3.4) in the form:

$$f(r, t) = \Sigma S_k(t) R_k(r) = -\sum_{k=1}^{\infty} R_k(r)\,\frac{\lambda_k^2\exp\left(-\lambda_k^2 t\right)}{B_k(t)}\int_0^t H_k(t)\exp\left(\lambda_k^2 t\right)dt \tag{3.12}$$

$$\left(B_k(t) = \int_{r_1}^{r_2}\left(\frac{2R_k}{r^2} - R_k''\right)^2 dr\right)$$

Putting (3.12) into (3.2), the stream function can also be expressed as a series.

We find the velocity components to the first approximation in Rayleigh numbers from the formulas

$$u_r = \frac{N_{\rm Pr} N_{\rm Gr}}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \qquad u_0 = -\frac{N_{\rm Pr} N_{\rm Gr}}{r \sin\theta} \frac{\partial\psi}{\partial r} \tag{3.13}$$

2. We will now turn our attention to finding the fluid temperature. The second equation in (1.9), which in the first approximation yields the convective portion of the temperature distribution, can be written in spherical coordinates [(bearing in mind (3.1) and (3.2)] in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tau_1}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\tau_1}{\partial\theta}\right) - N_{\rm Pr}\frac{\partial\tau_1}{\partial t} = 2\cos\theta \frac{f}{r^2}\frac{\partial\tau_0}{\partial r} \tag{3.14}$$

The initial and the boundary conditions of the problem (see section 1), also taking into account the conditions under which the zero approximation temperature equation (see section 2) was solved, give

$$\tau_1(r_1, \theta, t) = \tau_1(r_2, \theta, t) = 0, \qquad \tau_1(r, \theta, 0) = 0 \tag{3.15}$$

We will look for a solution of equation (3.14) in the form

$$\tau_1 = \varphi(r, t)\cos\theta \tag{3.16}$$

Now putting (3.16) into (3.14) we arrive at the equation which the function ϕ must satisfy

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\varphi}{\partial r}\right) - \frac{2\varphi}{r^2} - N_{\rm Pr}\frac{\partial\varphi}{\partial t} = \frac{2f}{r^2}\frac{\partial\tau_0}{\partial r} \tag{3.17}$$

From (3.15), we have the following conditions for the function ϕ:

$$\varphi(r_1, t) = \varphi(r_2, t) = 0, \qquad \varphi\ (r,\ 0 = 0) \tag{3.18}$$

Using the method of separation of variables we find the solution of the problem (3.17) and (3.18).

$$\varphi(r, t) = -\frac{1}{N_{Pr}} \sum_{j=1}^{\infty} D_i(r) \exp\left(-\frac{\varepsilon_i^2}{N_{Pr}}\, t\right) \int_0^t M_i(t) \exp\left(\frac{\varepsilon_i^2}{N_{\rm Pr}}\, t\right) dt \tag{3.19}$$

Here,

$$D_i(r) = (\varepsilon_i r)^{-1/2}\left[J_{-3/2}(\varepsilon_i r_1) J_{3/2}(\varepsilon_i r) - J_{3/2}(\varepsilon_i r_1) J_{-3/2}(\varepsilon_i r)\right] \tag{3.20}$$

are eigen functions of the homogeneous problem corresponding to the non-homogeneous problem (3.17)–(3.18), and ϵ_i are roots of the equation,

$$\operatorname{tg}\varepsilon = \frac{\varepsilon}{1 + r_1 r_2 \varepsilon^2}, \quad M_i(t) = \int_{r_1}^{r_2} 2f \frac{\partial\tau_0}{\partial r} D_i(r)\, dr \Bigg/ \int_{r_1}^{r_2} r^2 D_i^2(r)\, dr$$

Therefore in the first approximation we get for the dimensionless temperature

$$\tau = \tau_0 + N_{Pr} N_{Gr} \tau_1 \tag{3.21}$$

where τ_0 is obtained from formula (2.2), and τ_1 from formulas (3.16) and (3.1).

4. Heat Flow. Cooling Time. 1. We will now find the quantity of heat Q flowing through the boundary over a finite time interval t (for instance, over the time t from the start of the fluid cooling process).

For this, we make use of the formula

$$Q = -\lambda \int_0^t \int_{(s)} \left(\frac{\partial T}{\partial n}\right)_s ds\, dt \tag{4.1}$$

where λ is the coefficient of heat conductivity of the fluid, (s) is the bounding spherical surface of the film, n is the outward normal to this surface. In the approximation we are dealing with, this fluid temperature distribution is given by the formula

$$T = T_1 + (T_2 - T_1)\,(\tau_0 + N_{\rm Pr} N_{\rm Gr} \tau_1) \tag{4.2}$$

where τ_0 and τ_1 are determined from equations (2.2), (3.16) and (3.1). If we put (4.2) into (4.1) and integrate we find;

$$Q = -\frac{2\pi\lambda l^3}{\chi} (T_2 - T_1) \sum_{n=1}^{\infty} A_n^2 \left[\exp\left(-\frac{n^2\pi^2}{l^2}\chi t\right) - 1\right] \tag{4.3}$$

Equation (4.3) shows that the heat lost by the fluid in time increases asymptotically to the value $(\lambda/\chi) V(T_2 - T_1)$, where V is the volume of fluid.

2. In the study of transient heat transfer problems, the cooling time or "equalization" time is an important quantity.

If we study the solution we find that in the case of our first approximation, convection does not have much effect on the heat transfer of the fluid, i.e. molecular heat conduction is the main controlling factor in the heat transfer process.

Indeed, as may be seen from the solution of the problem, the temperature distribution in the fluid is determined by the sum of products of functions of the coordinates and exponential functions of time. Obviously the rapidity of temperature change is mainly determined by that term in the sum which has the least absolute value of the coefficient of t. The reciprocal of this coefficient can be used to express the "equalization" time. It appears that the temperature "equalization" time is given by $t = l^2/\chi\pi^2$ for both cases, i.e., for purely molecular thermal conduction, and for the case when thermal convection affects the temperature distribution.

We should be able to obtain a graphic picture of the heat transfer process if, in any plane containing the axis of symmetry of the flow, we drew out streamlines and isothermals for various instants of time. However this was not done because the formulas obtained are so unwieldy.

The author thanks I.G. Shapshnikov for making available the subject of the work, and S.I. Melnik and V.S. Sorokin for some valuable criticisms.

BIBLIOGRAPHY

1. Landau, L. and Lifshits, E., *Mekhanika sploshnykh sred* (*Mechanics of Continuous Media*). GITTL, 1954.

2. Shaposhnikov, I.G., K teorii slaboi konvektsii (Contribution to the theory of weak convection). *Zh. Tekh. Fiz.* Vol. 22, No. 5, p. 826, 1952.

3. Kamke, E., *Spravochnik po obyknovennym differentsial'nym uravneiniam* (*Guide to the Common Differential Equations*). IL, 1951.

4. Grinberg, G.A., Izbrannye voprosy matematicheskoi teorii electricheskikh i magnitnikh iavlenii (Some selected problems in the mathematical theory of electrical and magnetic problems). *Izv. Akad. Nauk SSSR* 1948.

Translated by V.H.B.